ECL 1002

Engineering Case Library

A Case History

A Critical Meeting on the Development of Inertial Navigation

-Related to "The Story of 84 Minutes"

by

William Bollay

Visiting Professor, Stanford University

ABSTRACT: In 1949 a team of engineers from North America Aviation informed the Secretary of the U.S. Air Force about the first successful tests of the first large U.S. developed rocket, and the successful laboratory tests of a new high precision inertial guidance system. The Secretary challenged the engineers to repeat the presentation in the presence of a distinguished scientist, who had concluded only a few weeks earlier that the Air Force had better stick with manned aircraft, because long range missiles would not be able to achieve the necessary guidance accuracy.

This case history presents the discussion which took place, including an explanation of the 84 minute pendulum effect, which reduced the target error for a 5000 mile cruise missile to less than 1/100 of the value expected by the simple theory.

This case study, including illustrative problems, is a supplement to Professor J. P. Den Hartog's Case No. 1001 on "The Story of 84 Minutes".

The illustrative problems bring out the fact that inertial navigation systems have interesting dynamic characteristics: in the x y plane any errors in acceleration measurement are counteracted by the erroneous tilt of the accelerometer platform. In the z direction these errors lead to a rapid divergence.

Prepared with the support of the National Science Foundation.

A Case History Related to Inertial Navigation--

"The Story of 84 Minutes"

by

William Bollay *

Introduction

One of the memorable experiences in my life was a briefing that I and my associates had to present in 1949 to Mr. Symington, the Secretary of the U.S. Air Force, on the status of the North American Aviation guided missile program. I was Technical Director of the Aerophysics Laboratory at North American Aviation and we had just completed our first successful tests of the large liquid fueled rockets and of our newly developed inertial guidance components. Mr. Symington listened to our story carefully and said "What you tell me about guidance accuracies does not agree with what I heard only a few days ago from one of America's most distinguished scientists, Dr. X. He tells me that a missile with a range of two thousand miles might hit within 150 miles of its target, but that for high accuracy delivery of

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* Now, Visiting Professor, Stanford University, teaching courses in Space Systems Engineering.

The other members of the NAA briefing team at this critical meeting included:

John L. Barnes	- Then, Chief of Guidance, NAA - Now, Professor of Engineering, University of California, Los Angeles
John R. Moore	- Group Leader, in charge of Inertial Guidance. Mr. J.R. Moore was the young engineer at NAA who prepared the first analysis of the 84 minute pendulum effect upon the accuracy of the NAA guidance system. - Later, President, Autonetics Division of NAA - Now, Executive Vice-President, North American Aviation, Inc.
Samuel K. Hoffman	- Then, Chief of Propulsion, NAA - Now, President, Rocketdyne Division of NAA

warheads we had better stick to manned aircraft. Yet you say that with your new guidance systems the errors would be less than a hundredth as much. Would you be willing to repeat to Dr. X what you have just said to me?"

We said we would be pleased to do so, that, in fact our chief of guidance had studied under Dr. X and that we believed he would confirm that our conclusions were correct.

We therefore presented our briefing again a few days later to Dr. X and the Secretary of the Air Force. Our spokesman said: "We should like to bring you up to date on some remarkable new developments in the field of inertial guidance. During the past three years we have undertaken an intensive research and development program to develop improved components for high precision inertial guidance systems. As you may remember, the V-2 had an accuracy of a couple of miles in a range of 200 miles. We have now developed guidance components which in the laboratory have an accuracy between 10 and 100 times as great as those of the V-2 vintage. We don't know yet how well we will be able to do in the flight environment of actual missiles, but we think that even if we degrade this laboratory performance by a factor of 2 we'll have acceptable performance to achieve high precision long range missiles."

Dr. X interposed as follows: "John, this is certainly a remarkable achievement. It seems to me, however, that even if you achieve a reduction of the error of a 200 mile missile from 2 miles to 0.2 miles this is still far from adequate. Mr. Newton's laws state that

$$s = \frac{1}{2} a t^2$$

Thus a given error in the accelerometer Δa results in a range error of

$$\Delta s = \frac{1}{2} \Delta a \cdot t^2$$

Thus, if you fly a missile for 10 times as long, the error in range will be 100 times as great. Moreover, I understand that the drift rate of the gyro causes a target error proportional to t^3 .

Thus the actual error might be between 10^2 and 10^3 as great as the 1/5 mile corresponding to a 200 mile missile."

Our guidance chief went on with his explanation: "What you say is correct for short range missiles when it is a good approximation to assume the flat earth equation

$s = \frac{1}{2}at^2$ etc. There is a very major reduction in this error which occurs for long flight times when you consider a circular trajectory over a spherical earth. This phenomenon which was discovered by a German physicist, M. Schuler, way back in 1923 is sometimes called "The 84 Minute Pendulum Effect." For example, this effect reduced the position error Δx due to an accelerometer error $\Delta a = k g$ to an amount

$$\Delta x = k R \cdot \left[1 - \cos \frac{2\pi t}{T}\right]$$

and at $t = \frac{T}{2}$ | where R = 4000 miles = radius of earth

$\Delta x_{max} = 2kR$ | T = 84.4 minutes

Thus the maximum error is $2kR$ independent of flight time instead of being proportional to t^2 like the simple theory would indicate. For an accelerometer accuracy $\Delta a = 10^{-4} g$ this maximum error is $\Delta x_{max} = 2 \times 10^{-4} \times 4000$ miles $= 0.8$ mile or about 4000 feet. With further development we believe there may be a chance of reducing even this error by another order of magnitude.

You are correct in stating that gyro drift is an even more serious source of error. The simple theory would indicate that if the gyro drifts slightly in the x-direction to an angle φ then the accelerometers would sense a component of gravity equal to φg in the same direction. If the gyro has a constant drift rate so that $\varphi = \varepsilon t$ then the velocity error becomes

$$\Delta V = \int_0^t \Delta a \cdot dt = \int_0^t \varepsilon t \cdot g dt = \frac{\varepsilon g t^2}{2}$$

and the position error

$$\Delta x = \int_0^t \Delta V \cdot dt = \frac{\varepsilon g t^3}{6}$$

This is the reason for the cubic relation between distance and time of flight. However, this relation again only applies to a flat earth. For a spherical earth we can show that the position error due to a constant gyro drift rate ε is only

$$\Delta x = \frac{R\varepsilon}{\omega}\left[\omega t - \sin \omega t\right]$$

where $\omega = \frac{2\pi}{T}$ and T is again this magical 84 minute period.

Thus for large values of time this error is only proportional to the first power of t instead of t^3 and it is this fact which makes inertial guidance a practical possibility.

During the World War II period the best flight gyros had a drift rate of about 3 degrees per 15 minutes or 12 degrees per hour.*) The V-2 gyros were better than this by about one order of magnitude. Our most recent gyro tests indicate a drift rate of the order of $\varepsilon = 0.01$ degrees per hour. If we can achieve a similar low drift rate in flight then over a period of t = 84 minutes = 1.4 hours the error due to gyro drift is about

$$\Delta x = R\varepsilon t \qquad \text{or} \quad \Delta x = 4000 \times \frac{0.01}{57.3} \times 1.4 \text{ miles}$$

$$\Delta x \approx 1 \text{ mile}$$

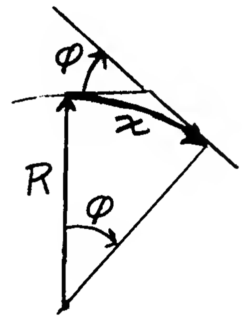
We can reduce this error further either by adding star trackers or, possibly, by further improvements of the gyros.

Dr. X said: "John, this is the most amazing story I have heard in a long time. There is one point you haven't explained and this is the basic mechanism of this mysterious 84 minute pendulum. How does it work and what is the significance of the 84 minute period? It seems to me I've heard talk that this is also the period of a low earth satellite."

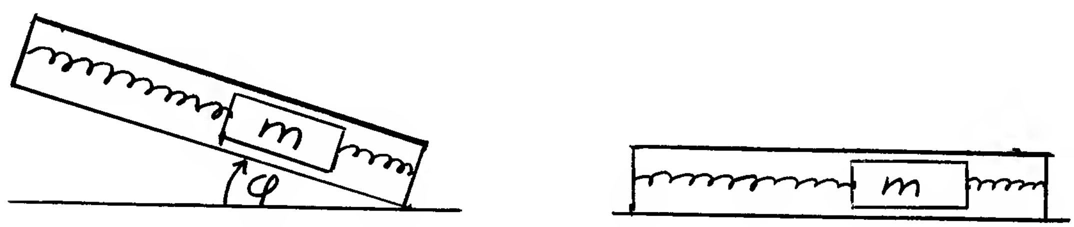
Our guidance chief went on with his explanation: "As a matter of fact, this 84 minute pendulum effect, or Schuler tuning as it is sometimes called, may be explained as a simple feed-back phenomenon such as you used to lecture about in your class. An inertial guidance system may be considered in principle as a doubly integrating accelerometer which is mounted on a level platform. The doubly integrating accelerometer is a distance meter and we have developed a very simple version of this which we shall explain a little later. If the vehicle travels in a circular trajectory the x-distance meter continuously cranks out on a revolution counter the distance travelled in the x-direction just like the mileage counter on your car. In principle, a shaft take-off from this distance meter could be connected to a worm gear which continuously levels the platform on which the distance meter is mounted. If the vehicle travels an arc distance x over the earth and R is the earth's radius the

*) See G.E. Irvin--Aircraft Instrument--McGraw Hill Book Co. (1941)

change in the angle of the level position is $\varphi = \frac{x}{R}$, The gear ratio between the x position meter and the angle φ is set to meet this condition.



Now consider what happens when you have an accelerometer which reads too high so that Δa is a positive number. To keep the picture simple, just assume that we have a guidance platform standing still but we are feeding this erroneous acceleration $\Delta a = k g$ into the doubly integrating accelerometer. The distance meter then cranks out a displacement x and this output shaft now cranks the platform to an angle $\varphi = \frac{x}{R}$ just as though the platform had been moved over the earth by the distance x. You may consider the accelerometer



Accelerometer at angle φ is equivalent to Accelerometer accelerated to the left.

in principle as a mass mounted between two springs. When the accelerometer is at an angle φ it is the same as though the accelerometer had been accelerated leftward with an acceleration φg .

Thus the apparent acceleration on this system is reduced by this feed-back to

$$\ddot{x} = kg - \varphi g$$

and since $\varphi = \frac{x}{R}$ we have $\ddot{x} = kg - \frac{g}{R} x$

or $$\ddot{x} + \frac{g}{R} x = kg$$

which has the solution

$$x = kR\left[1 - \cos \omega t\right]$$

where $\omega = \sqrt{\frac{g}{R}} = \frac{2\pi}{T}$ where $T = \frac{2\pi}{\sqrt{\frac{g}{R}}}$ = 84.4 minutes.

The solution for a gyro drift is exactly similar. If the gyro drift is $\varphi_1 = \varepsilon t$ then the reference platform drifts at this rate with respect to the true horizontal.

Thus an erroneous acceleration $\varphi_1 g$ is sensed by the distance meter and it cranks out an angular correction $-\frac{x}{R}$.

Thus the resultant angle of the platform is $\varphi = \varphi_1 - \frac{x}{R} = \varepsilon t - \frac{x}{R}$

and the apparent acceleration is φg or

$$\ddot{x} = \left(\varepsilon t - \frac{x}{R}\right) g$$

or

$$\ddot{x} + \frac{g}{R} x = \varepsilon g t$$

with a solution

$$x = \frac{R\varepsilon}{\omega}\left[\omega t - \sin \omega t\right]$$

where $\omega = \sqrt{\frac{g}{R}} = \frac{2\pi}{T}$

and T = 84.4 minutes

Thus, you will note that this error due to gyro drift only grows proportional to t for large values of t while for small values of t, as you indicated, it is proportional to t^3."

Dr. X shook his head, and said "You've given me a lot of food for thought. I would not have thought it possible that such inertial systems stood any chance of reaching high precision. We'll have to see what you can accomplish with your flight tests during the coming years."

The meeting adjourned. Shortly thereafter the Air Force asked us to reorient our missile program toward a longer range system, toward what was later known as the Navaho program.

The Navaho was a ram jet powered missile cruising at a constant Mach number of 2.75. This missile was designed to be boosted by a large liquid propellant rocket. We made this decision for a ram jet powered cruise missile in 1949 because the real objective of the U.S. Air Force was to obtain a guidance accuracy at least one order of magnitude greater than even our very best accelerometers and gyros had demonstrated at that time. By using a cruising missile we had the opportunity of adding star trackers to the inertial guidance system and thus gain this additional factor of 10 in accuracy.

What has happened to the Navaho? An intermediate range version of it, powered by turbo-jets and later called the X-10 demonstrated successful flight at Mach numbers above 2. The ram jet powered Navaho became obsolete before it was fully developed. In the early 1950s thermo-nuclear warheads were developed which did not require the extreme guidance accuracy specified for the Navaho. It was therefore decided to concentrate all of the ICBM effort on the ballistic type of missile, which had the additional advantage of being more difficult to intercept.

Ballistic missiles may also be guided by inertial systems during the boost phase. Since this boost phase lasts only for about three minutes, the 84 minute pendulum effect is of less importance for ballistic missile guidance.*) The same requirements exist, however, for high precision accelerometers and accurate alignment of the accelerometers. (See Fig 1, page 8.)

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*) For a ballistic missile the period T of the Schuler pendulum is not 84 minutes, since $T = 2\pi\sqrt{\frac{R}{g_{eff}}}$ where g_{eff} is the effective vertical acceleration experienced by the accelerometer case. Let us examine two limiting cases:

a) If the missile is accelerating vertically with a thrust equal to n times the weight, then $g_{eff} = n\, g_o$ where $g_o = 32.2\ \frac{ft}{sec^2}$

For a vertical trajectory with a vertical acceleration $g_{eff} = 9\, g_o$ or $n = 9$ and $T = \frac{T_o}{\sqrt{n}} = \frac{84}{\sqrt{9}} = 28$ minutes.

b) For a missile flying at velocity V parallel to the earth $g_{eff} = g - \frac{V^2}{r}$ where r is the radius of curvature of the trajectory. Thus g_{eff} approaches zero as the velocity V approaches the satellite velocity V_c and the period T of the Schuler pendulum approaches ∞.

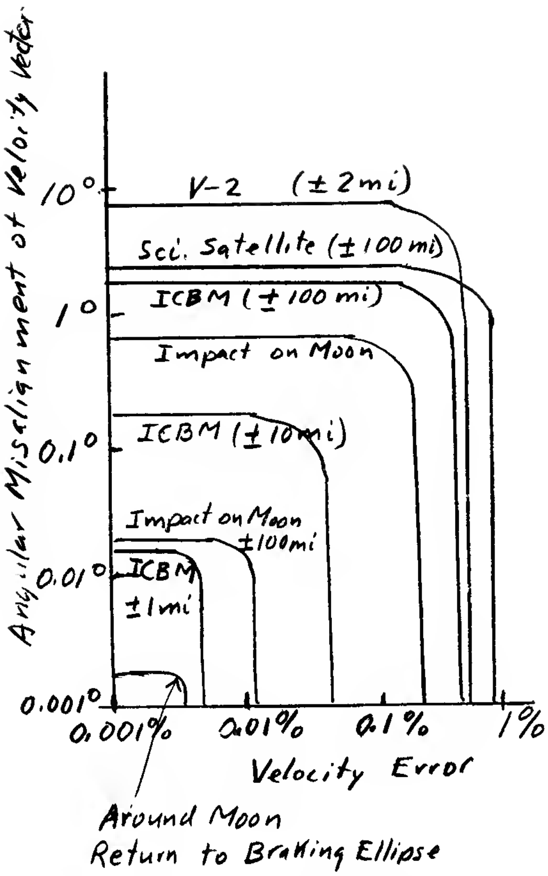

Figure 1

Reference: Fundamentals of Missile Guidance by John R. Moore and Charles P. Greening, Autonetics, A Div. of North American Aviation. - Astronautics, May 1958, page 24.

The Navaho rocket system was redesigned to fit the Atlas ICBM. The Navaho inertial guidance system later became the basis of the inertial navigation for submarines, ships, and long range aircraft. For example, the nuclear powered submarine cruised under the Polar Ice Cap guided by the NAA Inertial Navigation System. Similar inertial guidance systems are currently being installed in long range transport airplanes. Different versions of inertial guidance systems are being used for rendezvous of space craft in orbit, and will be used for the self-contained guidance of the Saturn V rocket booster to the moon. These new uses of inertial guidance systems are made feasible by the continued growth in accuracy of the accelerometers and gyros. We made a gain in gyro and accelerometer accuracies by about two orders of magnitude between 1945 and 1949. During the past 17 years even further improvements have been made. The precise state of the art is still classified.

MORAL

The moral of this story might be expressed in terms of CLARKE'S LAW:

> When a distinguished but elderly scientist states that something is possible, he is almost certainly right. When he states that something is impossible, he is very probably wrong.

For further evidence of this lesson, the reader is encouraged to read pages 1-21, reference (10).

References:

(1) M. Schuler - The Disturbance of Pendulum and Gyroscopic Apparatus by the Acceleration of the Vehicle. (In German) Journal of Physics, Vol. 24, July 1923. Translation to English by J.M. Slater, Autonetics Report, dated 1 Oct. 1960.

(2) J.N. Slater - Inertial Guidance Sensors- Reinhold Publishing Co. (1964)

(3) C.F. O'Donnell- Inertial Navigation, Analysis and Design, McGraw Hill Book Co. (1964)

(4) Howard Savant, Solloway and Savant - Principles of Inertial Navigation, McGraw Hill Book Co. (1961)

(5) C.S. Draper, W. Wrigley and J. Hovorka - Inertial Guidance, Pergamun Press (1960)

The following unclassified reports on inertial navigation systems are available to professors of U.S. Universities who request such a copy for class reference use from:

Autonetics
A Division of North American Aviation Inc.
3370 Miraloma Avenue
Anaheim, California 92803

(6) J.M. Slater - Newtonian Navigation

(7) J.M. Slater - Daylight Star Tracking

(8) J.M. Slater - Horizon Scanning and Planet Tracking

(9) Light Weight N-16 Autonavigator-- Publ. P4-903/32

The following popular book by a well known science fiction writer may be of interest to the students with respect to the subject of technical forecasts:

(10) Arthur C. Clarke -Profiles of the Future, Bantam Pathfinders Editions HP 123 ($0.60)

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COPY

Stuart Symington
Missouri

United States Senate

Washington, D.C. 20510

November 25, 1966

Professor William Bollay
Mechanical Engineering Department
Stanford University
Stanford, California

Dear Professor Bollay:

Acknowledging your letter of November 7, I well remember it, also remember getting in touch with Dutch Kindelberger immediately after the briefing. You all stayed over an extra day to do it for me, and I think it had as much as anything to do with keeping missile development on the rails.

Thanks for "Profiles of the Future." I will read it next week on my way to the Far East.

As to the rest of your comments, I am not enough of an engineer to make an adequate analysis; but I do know that that briefing, plus the second briefing, changed the thinking, as contained in that famous quotation, of Dr. X; and that in turn guaranteed the future of the guided missile.

Sincerely,

Stuart Symington

SS:ag

Problem 1: Using the flat earth approximation, find the error of an inertial guidance system due to the following sources:

(a) Error in initial platform angular alignment

$$\Delta \theta_y = 0.01 \text{ degree}$$

(b) Error in setting initial launch velocity $\Delta V_x = \Delta V_y = \frac{1\ FT}{SEC}$ foot per second due to launching from a moving platform (ship, submarine, or airplane).

(c) Error in accelerometers

$$\Delta a = 10^{-4} g$$

(d) Error due to gyro drift

$$\epsilon = 0.01 \frac{\text{degree}}{\text{hour}}$$

Problem 2: Repeat Problem 1 assuming a circular trajectory over a spherical earth.

Problem 3: Compute a numerical example for problems (1) and (2) for a missile flying a distance of 5,060 miles for which the flight time t = 2 x 84.4 minutes = 10, 120 sec. so that $\omega t = \frac{2\pi t}{T} = 4\pi$ assuming a flying speed of 0.5 miles per second. Plot each of these errors as a function of time.

Problem 4: Study the feasibility of using an inertial guidance system to keep a helicopter hovering accurately at a fixed height, i.e., using the inertial guidance in the vertical z direction. For simplicity consider only one dimensional motion in the vertical direction.

Teaching note

This case history may be used in conjunction with J.P. Den Hartog's Case "The Story of 84 Minutes," Section IX, to present the principles of inertial guidance. This new technology is based upon principles discovered by M. Schuler in 1923. (Ref. 1). A simple description of the devices--the accelerometers, gyros, and inertial platforms--is given in Ref. (6). A very complete description including mathematical analysis, diagrams, and pictures of devices is given in Ref. (3).

It is believed that this case might be suited as source material for an introductory physics course to illustrate the principles of inertial guidance. A freshman physics student should be able to solve Problem (1), and understand how tilting the accelerometer results in a restoring effect on the inertial guidance system. He could prove by differentiation that the quoted solutions, equations (2) and (4), solve the equations of motion (1) and (3) respectively.

A junior or senior student in dynamics should be able to solve Problems (1) or (4) for which the solutions are attached. Problem (4) shows that inertial devices can not be used to determine altitude because any small errors in accelerometer measurement lead to a rapid divergence in vertical position indication.

It may be encouraging to the beginning student of engineering to contemplate the vast opportunities for new developments which are opened as a result of technological break-throughs such as the improvement of accelerometer and gyro accuracies. Reference (10) presents a science fiction writer's views on this subject. The students might be encouraged in some of their technical writing courses to use their imaginations and project what might happen whenever an order of magnitude gain is accomplished. For example, what would be the effect of a reduction in the weight/power ratio of batteries or fuel cells upon our cities and our economy? (According to the New Scientist--13 October 1966, page 7--a new battery developed by Ford will increase the energy content from 8-10 watt-hours per pound (for the lead-acid battery) to 150 watt-hours per pound (for the new sodium-sulphur battery).)

Solution: Problems 1, 2, and 3:

(a) Error due to initial platform misalignment

$$\Delta\theta_y = 0.01 \text{ degree}$$

Flat Earth Approximation: *(values at $t = 10{,}120$ sec)*

Misalignment $\Delta\theta_y$ = .01 degree = 1.74×10^{-4} *radians*

Acceleration error $\Delta a_x = g \cdot \Delta\theta_y = 1.74 \times 10^{-4} g = 0.0056 \frac{ft}{sec^2}$

Velocity error $\Delta V_y = \int_0^t \Delta a_x \cdot dt = g \cdot \Delta\theta_y \cdot t = 56 t \times 10^{-4} \frac{ft}{sec}$ or $57 \frac{ft}{sec}$

Position error $\Delta s_x = \int_0^t \Delta V_x \cdot dt = g \cdot \Delta\theta_y \cdot \frac{t^2}{2} = 28 t^2 \times 10^{-4} ft = 287{,}500\ ft$

The errors due to an initial platform misalignment $\Delta\theta_x$ = .01 degree are exactly similar, i.e.

$$\Delta s_y = g \cdot \Delta\theta_x \cdot \frac{t^2}{2} = 28 t^2 \times 10^{-4} ft = 287{,}500\ ft$$

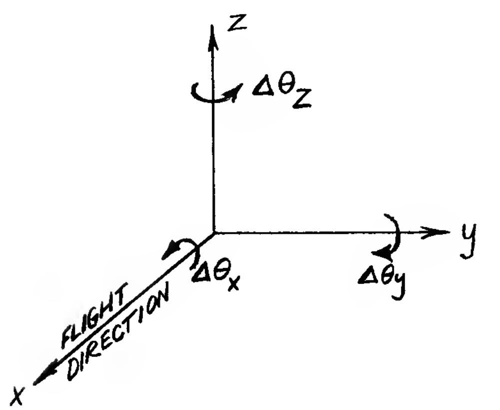
The error due to a misalignment in azimuth, i.e., about the vertical axis is simply equal to the product of range $\times\ \Delta\theta_z$. Thus for a range of 5, 060 miles, the position error due to $\Delta\theta_z$ is

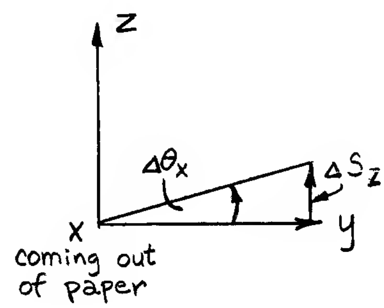
$$\Delta s_y = 5060 \times \frac{.01}{57.3} \text{ mile} = 4{,}660 \text{ ft}$$

Note: $\Delta\theta_x$ = small rotation about the x-axis

$\Delta\theta_y$ = small rotation about the y-axis

$\Delta\theta_z$ = small rotation about the z-axis





Spherical Earth Approximation:

From Den Hartog's Case Study

$$\Delta S_x = R \cdot \Delta\theta_y \left[1 - \cos \omega t\right]$$

where R = earth radius = 3960 statute miles

$$\omega = \frac{2\pi}{T}$$

T = 84.4 minutes = 5060 seconds,

The maximum error at $\omega t = \pi$ is

$$\Delta S_x = 2R \cdot \Delta\theta_y = 2 \times 3960 \times \frac{0.01}{57.3} \text{ miles} = 1.38 \text{ miles} = 7300 ft.$$

In general, the error is

$$\Delta S_x = 3650 \, ft \left[1 - \cos \frac{2\pi t}{T}\right].$$

For the case t = 10,120 sec., $\omega t = \frac{2\pi t}{5060} = 4\pi$

and $\cos \omega t = 1$. Thus $\Delta S_x = 0.$

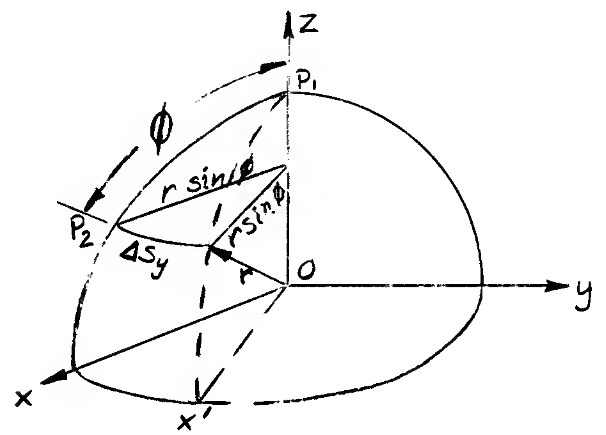
The error due to misalignment $\Delta\theta_y$ thus varies as shown in the figure below:

The error Δs_y follows an exactly similar relation as Δs_x, i.e.

$$\Delta s_y = R \cdot \Delta\theta_x \cdot \left[1 - \cos \omega t\right] = 3650 \cdot \left[1 - \cos \frac{2\pi t}{5060}\right] ft.$$

However, the error due to a misalignment in azimuth $\Delta\theta_z$ does <u>not</u> follow the 84 minute pendulum relation. Assume the missile is launched at the point P_1 into the x' direction instead of the x direction. If the target point is P_2 at an angle ϕ around the earth so that the range is $r = R\phi$. Then the distance of P_2 from the axis OP_1 is $R \sin\phi$, and the lateral error due to an azimuth alignment

$$\Delta s_y = R \cdot \sin\phi \cdot \Delta\theta_z .$$



In our example $r = R\phi = 5060$ st. miles

Thus $\phi = \dfrac{5060}{3960} = 1.28$ radians $= 73.3^{\circ}$

$\sin = .957$

$R = 3960$ st. miles

and $\Delta s_y = 3960 \times .957 \times \dfrac{.01}{57.3}$
$= 0.662$ st. miles $= 3500$ ft

(b) Effect of error in measuring initial launch velocity
$\Delta V_x = \Delta V_y = 1 \frac{ft}{sec}$

Flat Earth Approximation:

The velocity at any time t is

$$\int_0^t dV_x = \int_0^t a_x \cdot dt$$

$$V_x(t) - V_x(0) = \int_0^t a_x \cdot dt$$

$$\therefore\ V_x = V_x(0) + \int_0^t a_x \cdot dt$$

and $$s(x) = \int_0^t V_x(t)\cdot dt = \underbrace{\int_0^t V_x(0)\cdot dt}_{V_x(0)\cdot t} + \int_0^t dt \int_0^t a_x \cdot dt$$

An error $\Delta V_x(0)$ in measuring $V_x(0)$ results therefore in a position error $\Delta s_x = \Delta V_x(0)\cdot t$

Thus, for t = 10,120 sec. and $\Delta V_x(0) = 1 \frac{ft}{sec}$

the error is Δs_x = 10,120 ft.

Similarly, an error $\Delta V_y(0) = 1 \frac{ft}{sec}$ would lead to an error Δs_y = 10,120 ft.

Spherical Earth Approximation:

Due to an initial velocity error $\Delta V_x(0)$ the platform cranks out an indicated distance Δs_x and thus assumes an angle $\phi = \frac{\Delta s_x}{R}$ to the horizontal. This results in an apparent acceleration

$$\Delta \ddot{s}_x = -\phi g = -\frac{g}{R}\cdot \Delta s_x .$$

Thus the equation of motion is

$$\Delta \ddot{s}_x + \frac{g}{R}\Delta s_x = 0$$

with the boundary conditions that at t = 0,

$$\Delta s_x = 0 \qquad and \qquad \frac{d}{dt}\Delta s_x = \Delta V_x(0).$$

This solution is $\Delta s_x = \frac{\Delta V_x(0)}{\omega} \cdot \sin \omega t$ where $\omega = \frac{2\pi}{T}$

and T = 5060 sec

and $$\frac{d}{dt}\Delta s_x = \Delta V_x(0)\cdot \cos \omega t$$

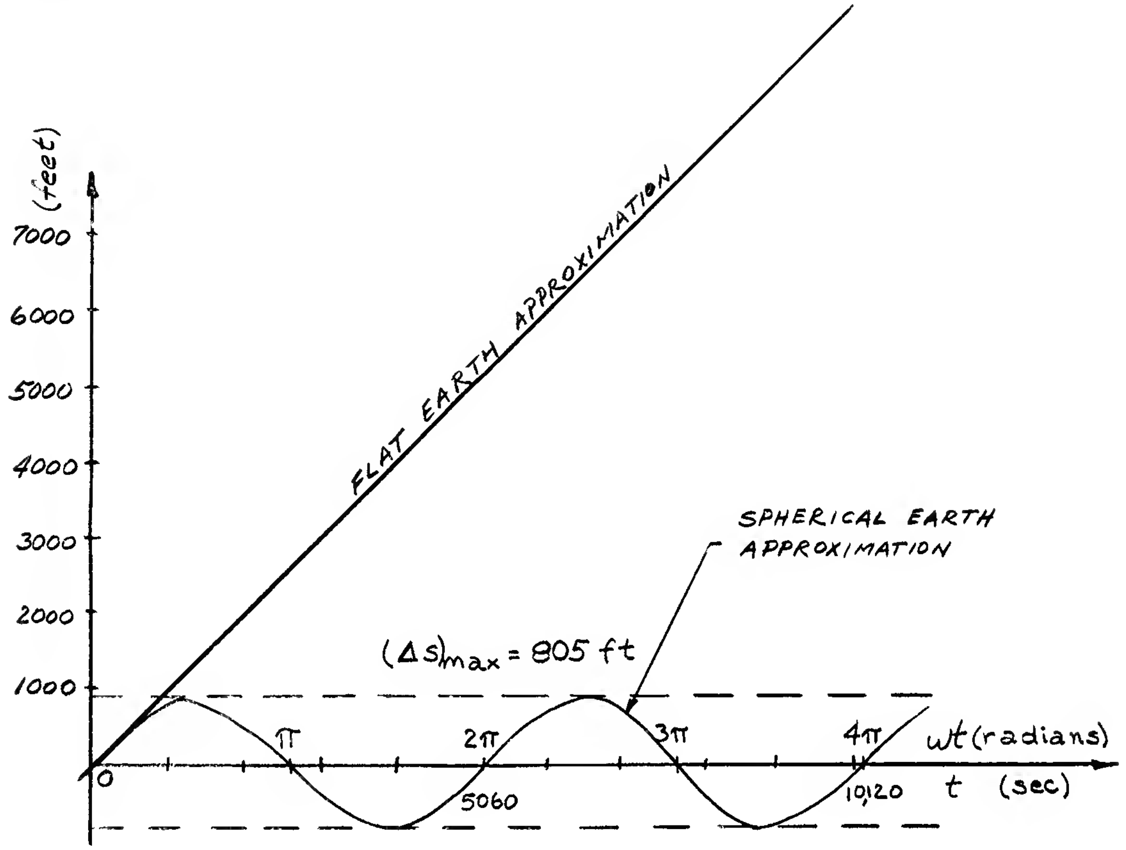
Thus the maximum error is reduced by the 84 minute pendulum effect to

$$(\Delta s_x)_{max} = \frac{\Delta V_x(0)}{\omega} = \frac{1}{\frac{2\pi}{5060}} = 805\,ft$$

and at $\omega t = 0, \pi, 2\pi, 3\pi, 4\pi, etc.,$ this error vanishes.

Similarly, the distance error due to an initial velocity error $\Delta V_y(0)$ is $\Delta s_y = \frac{\Delta V_y(0)}{\omega}\cdot \sin \omega t$.

A comparison of the flat earth and spherical earth approximations is shown below:



ASSUMPTIONS:

$\Delta V_x(0) = 1 \ \frac{ft}{sec}$

$\Delta s_y = \Delta s_x$

$\Delta V_y(0) = 1 \ \frac{ft}{sec}$

(c) Error due to accelerometers $\Delta a = kg = 10^{-4} g$

For the flat earth case, from Bollay Case Study, for $t = 10{,}120$ sec

$$\Delta s_x = \Delta a \cdot \frac{t^2}{2} = kg \cdot \frac{t^2}{2} \quad \text{or} \quad \Delta s_x = 10^{-4} \times 32.2 \times 10.12^2 \times 10^6 ft = 341{,}000 ft$$

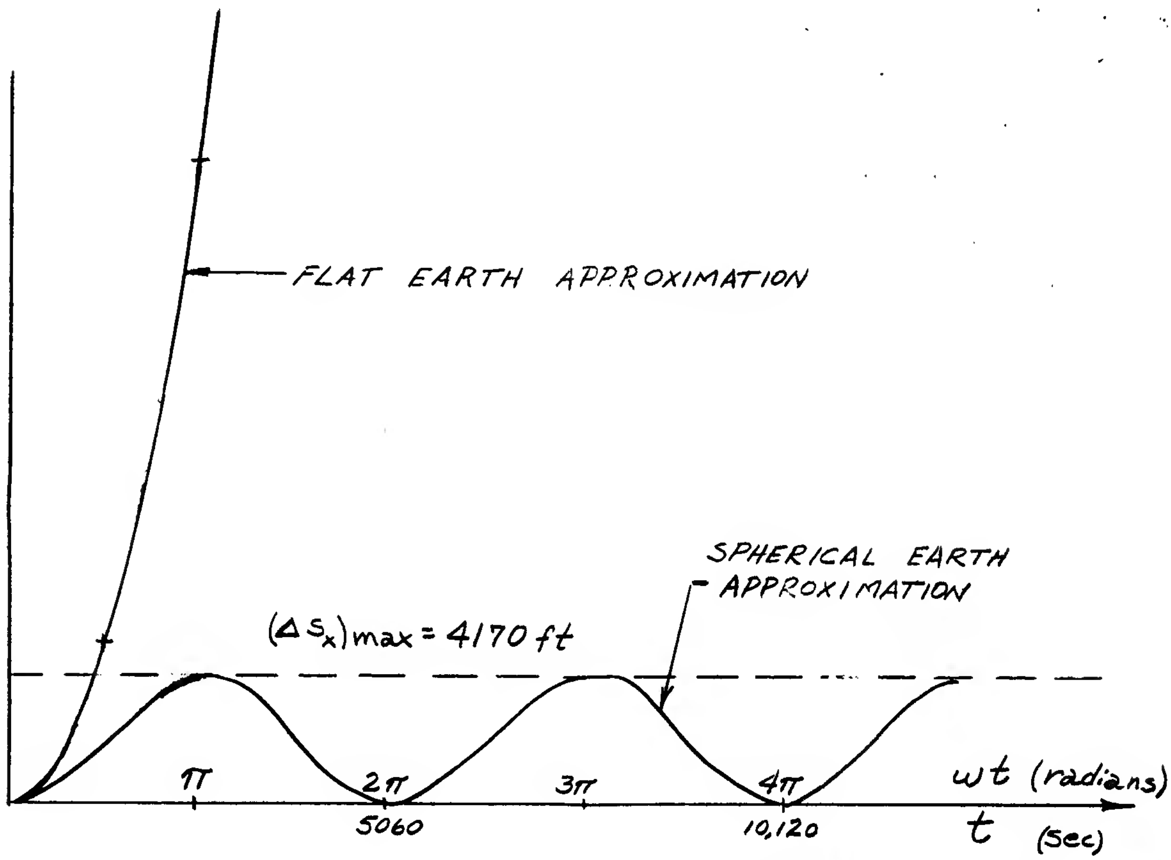
For the spherical earth case, from Bollay Case Study

$$\Delta s_x = kR \cdot [1 - \cos \omega t].$$

The maximum error is $(\Delta s_x)_{max} = 2kR$ at $\omega t = \pi, 3\pi$, etc.

where $(\Delta s_x)_{max} = 2 \times 10^{-4} \times 3960 \text{ miles} = 0.792 \text{ miles} = 4170 \text{ ft}$

At $\omega t = 0, 2\pi, 4\pi$, etc., this error vanishes



(d) Error due to gyro drift $\dot{\theta}_y = \varepsilon = 0.01 \frac{\text{degrees}}{\text{hour}} = \frac{0.01}{57.3 \times 3600} \frac{\text{rad}}{\text{sec}}$

Flat earth approximation:

Angular position $\theta_y = \varepsilon t$

Acceleration error $\Delta a_x = -\theta_y \cdot g = -\varepsilon g t$

Velocity error $\Delta V_x = \int_0^t \Delta a_x \cdot dt = -\frac{\varepsilon g t^2}{2}$

Position error $\Delta s_x = \int_0^t \Delta V_x \cdot dt = -\dfrac{\varepsilon g t^3}{6}$.

At t - 10,120 sec. this error could be

$$\Delta s_x = \frac{0.01 \times 32.2 \times 10.12^3 \times 10^9}{57.3 \times 3600} = 1.62 \times 10^6 \text{ ft}$$

Similarly, due to a gyro drift $\dot{\theta}_x = .01 \dfrac{\text{degrees}}{\text{hour}}$ the error in Δ_y would be

$$\Delta s_y = 1.62 \times 10^6 \text{ ft}.$$

Spherical Earth Approximation:

From the Bollay Case Study:

$$\Delta s_x = -\frac{R\varepsilon}{\omega}[\omega t - \sin \omega t].$$

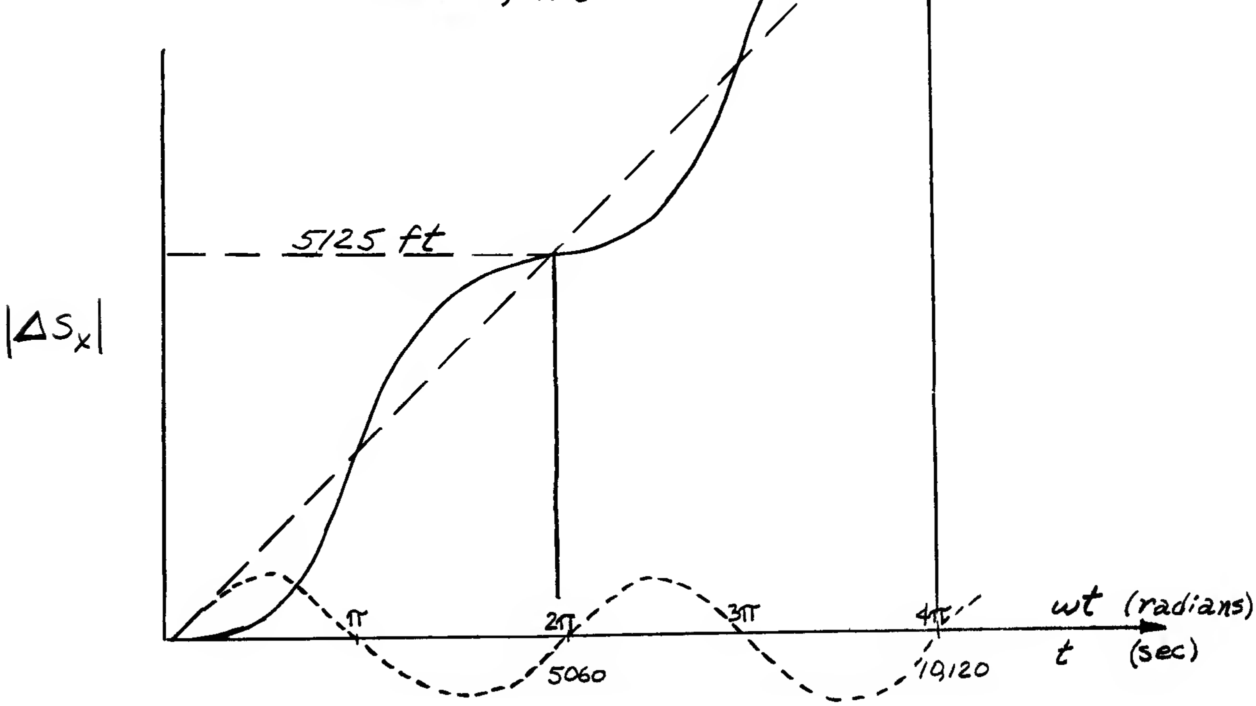
at $\omega t = 4\pi$ where t = 10,120 sec. the term $\sin \omega t = 0$ and thus $\Delta s_x = R\varepsilon t = 3960 \dfrac{.01}{57.3 \times 3600} \times 10{,}120$ miles

= 1.94 miles = 10,250 ft.

Thus the 84 minute pendulum effect has reduced the error of this term by a factor of more than 100.

The maximum platform error angle after 2.8 hours is only $\theta = \varepsilon t = 0.028$ degrees

$= 1.68$ minutes of arc



Summary and Conclusions:

(1) The 84 minute pendulum effect has reduced the error of inertial guidance systems compared to the flat earth calculations as follows:

	Flat Earth	Spherical Earth
Initial Misalignment in Level: $\Delta\theta_y = .01$ degree	$\Delta S_x = g \cdot \Delta\theta_y \cdot \frac{t^2}{2}$ at $t = 10{,}120$ sec $\Delta S_x = \underline{288{,}000\ ft}$	$\Delta S_x = R \cdot \Delta\theta_y \cdot [1-\cos\omega t]$ at $t = 10{,}120$ sec $\Delta S_x = 0$ Max. error occurs at $\omega t = \pi$ $(\Delta S_x)_{max} = \underline{3650\ ft}$
Launch Velocity Error: $\Delta V_x(0) = 1\ \frac{ft}{sec}$	$\Delta S_x = \Delta V_x(0) \cdot t$ at $t = 10{,}120$ sec $\Delta S_x = \underline{10{,}120\ ft}$	$\Delta S_x = \Delta V_x(0) \cdot \sin\omega t$ at $t = 10{,}120$ sec $\Delta S_x = 0$ Max. error occurs at $\omega t = \pi/2, 3\pi/2 \ldots$ $(\Delta S_x)_{max} = \underline{805\ ft}$
Accelerometer Error: $\Delta a = kg = 10^{-4} g$	$\Delta S_x = \frac{kgt^2}{2}$ at $t = 10{,}120$ sec $\Delta S_x = \underline{341{,}000\ ft}$	$\Delta S_x = kR[1-\cos\omega t]$ at $t = 10{,}120$ sec $\Delta S_x = 0$ Max. error occurs at $\omega t = \pi, 3\pi \ldots$ $(\Delta S_x)_{max} = \underline{4{,}170\ ft}$
Gyro Drift Error: $\dot{\theta}_y = \Sigma = .01\ \frac{degree}{hour}$	$\Delta S_x = \frac{\Sigma g t^3}{6}$ at $t = 10{,}120$ sec $\Delta S_x = \underline{1.62 \times 10^6\ ft}$	$\Delta S_x = \frac{R\Sigma}{\omega}[\omega t - \sin\omega t]$ at $t = 10{,}120$ sec $\Delta S_x = \underline{10{,}250\ ft}$
Azimuth Misalignment: $\Delta\theta_z = .01$ degree	$\Delta S_y = RANGE \times \frac{.01}{57.3}$ at $t = 10{,}120$ sec RANGE = 5060 miles $\Delta S_y = \underline{4660\ ft}$	$\Delta S_y = R \cdot \sin\phi \cdot \Delta\theta_z$ at $t = 10{,}120$ sec RANGE = 5060 miles $\Delta S_y = \underline{3500\ ft}$

Solution to Problem 4:

Let us consider an accelerometer which has been calibrated on a horizontal test track to read the true acceleration a. When it is used to read the vertical acceleration it therefore reads the resultant acceleration including that of gravity, i.e.,

$$a_z = \ddot{z} + g$$

or

$$\ddot{z} = a_z - g$$

where the acceleration of gravity g decreases with height according to the relation

$$g = \frac{g_0}{(1+\frac{h}{R})^2} \approx g_0\left[1 - \frac{2h}{R} + \cdots\right].$$

It can readily be seen that the use of an accelerometer for vertical position determination would lead to an unstable situation.

Consider a pilot in a vehicle which is in a hovering condition. The pilot has a vertical accelerometer which reads the true net acceleration. When this accelerometer is on the surface of the earth it reads the resultant acceleration of g_0. If it hovers at an altitude of h feet above the surface of the earth then it reads

$$a_z = \frac{g_0}{(1+\frac{h}{R})^2} \approx g_0\left[1 - \frac{2h}{R} + \cdots\right].$$

Thus, at a height of 2 miles it reads

$$a_z = g_0\left[1 - \frac{4}{4000}\right] = g_0\left[1 - \frac{1}{1000}\right]$$

Now let us assume the pilot is asked to hover at a height h, but his accelerometer reads <u>low</u> by an amount $|\Delta a_z| = kg_0$ so that $\Delta a_z = -kg_0$.

It reads $a_{zi} = a_z + \Delta a_z = a_z - kg_0$. When his indicated acceleration a_{zi} is held accurately at $a_{zi} = g_0[1 - \frac{2h}{R}]$, the vehicle is actually accelerating upward at the rate kg_0. After a time Δt he is at a height h + z, although he still thinks he is at height h. At this new height gravity has been reduced to

$$g = \frac{g_0}{\left[1 + \frac{h+z}{R}\right]^2} = g_0\left[1 - \frac{2h}{R} - \frac{2z}{R}\right]$$

He continues to apply power such that

$$a_{zi} = g_0\left[1 - \frac{2h}{R}\right].$$

Thus, he now experiences an additional net upward acceleration equal to

$$\Delta g = g_0 \cdot \frac{2z}{R} .$$

The motion of his vehicle may therefore be represented by the equation

$$\ddot{z} = \Delta a_z + \Delta g \qquad \text{where} \qquad \Delta a_z = -k g_0$$

$$\Delta g = g_0 \cdot \frac{2z}{R} .$$

Thus

$$\ddot{z} = -k g_0 + g_0 \cdot \frac{2z}{R}$$

or

$$\ddot{z} - 2\omega_0^2 z = -k g_0 \qquad \text{where} \qquad \omega_0^2 = \frac{g_0}{R}$$

This equation is similar to the 84 minute pendulum equation except for the factor 2 and that the sign of the second term is negative. The solution is

$$z = \frac{k g_0}{2\omega_0^2} \cdot \sinh\left(\sqrt{2}\,\omega_0 t\right) = \frac{kR}{2} \cdot \sinh\left(\sqrt{2}\,\frac{2\pi t}{T}\right) \text{ where } T = 84 \text{ min.}$$

Let us consider the error after a time t = 84 minutes where $\frac{t}{T} = 1$ and assume that $\Delta a = -k g_0$ where $k = 10^{-4}$.

Then $$z = \frac{10^{-4} \times 3960}{2} \sinh\left(\sqrt{2} \cdot 2\pi\right) = 724 \text{ miles.}$$

Thus we conclude that the inertial system cannot be used for determining vertical position on the earth, and that some other altitude measuring device (barometer, or radar altimeter) is necessary.